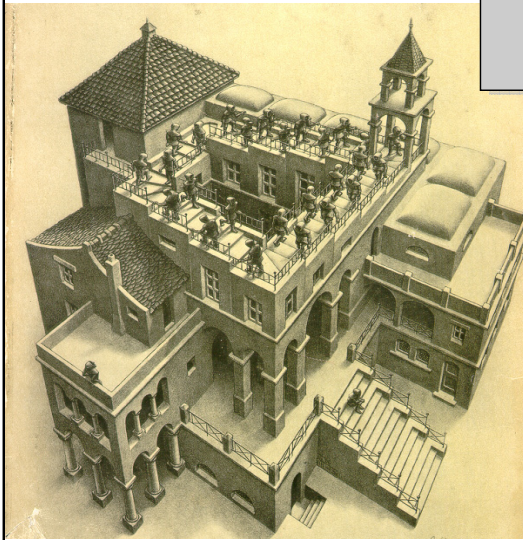


Nachrichtenverkehrstheorie Kap.3.1

- Teletraffic Engineering Basics-



Universität Hannover
Institut für Allgemeine Nachrichtentechnik
Kommunikationsnetze
Prof. Dr.-Ing. K. Jobmann

13.1.1

Literatur:

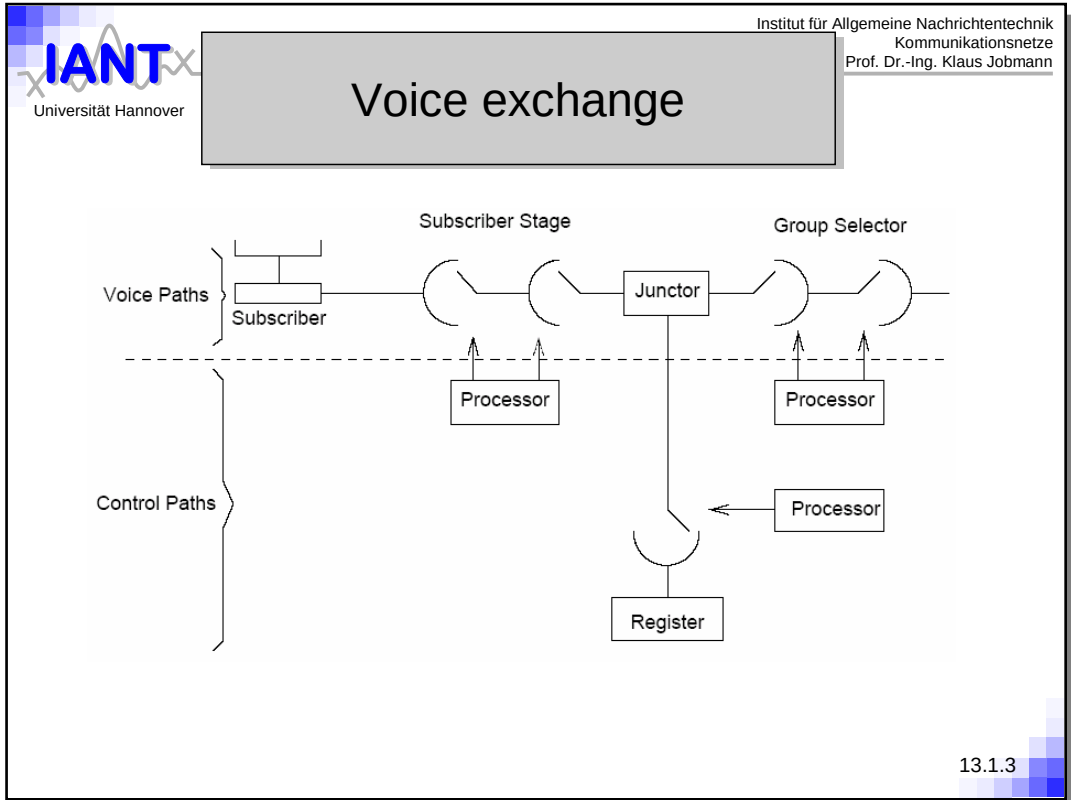
Mischa Schwartz, Telecommunication Networks, Protocols, Modelling and Analysis, Addison Wesley Publishing Company, 1988, ISBN 0-201-16423-X

Leonard Kleinrock, Queuing Systems, Volume I, Volume II, John Wiley & Sons, New York.,1975, ISBN 0-471-49110-1

ITU-D Study Group 2, „Handbook Teletraffic Engineering“, Geneva, December 2002

Outline

- **Teletraffic Concepts**
- Quality of Service – Grade of Service
- Erlang B Formula characterisation and further analysis
- Dimensioning of loss systems
- Loss systems with full accessibility
 - ⇒ Binomial case
 - ⇒ Engset case

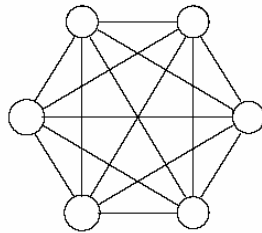


The figure shows a rather old-fashioned switching fabric, based on electromechanical technology. As can be seen, there are dialers in the voice path and there are separate control elements which make sure that the call is relayed from the input port to the correct output port. Since the control elements are occupied only during call setup (approx. 1s), whereas the voice path is occupied for the whole conversation (approx. 3 minutes on average), there has to be a significantly higher number of voice paths than control elements.

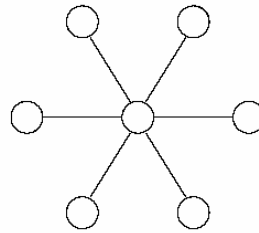
The voice path is usually designed as a loss system, whereas the control path is a waiting system.

Telephone network structure

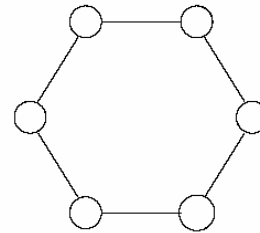
- Local exchanges (access level) are concentrated at transit exchange level (tree or star structure)
- Transit (or inter-urban) exchanges are usually fully meshed
- Some local exchanges may be connected if sufficient traffic



Mesh network



Star network



Ring network

13.1.4

Ring networks are usually build for reasons of availability, application usually in metropolitan area.
Examples: Optical ring networks, cable TV distribution

Traffic and its units

- Traffic = Traffic intensity (usually)
- Traffic intensity = Number of busy resources at a given time: $n(t)$
- Mean traffic intensity over a time period T : $Y(T)$

$$Y(T) = \frac{1}{T} \int_0^T n(t) \cdot dt$$

- Carried traffic A_c : The traffic served successfully (previously: throughput); usually equal to traffic intensity Y
- Offered traffic: Theoretical limit of carried traffic, if an infinite number of resources were available:

$$A = \lambda \cdot \frac{1}{\mu}$$

Traffic intensity = Verkehrswert

Carried traffic = getragener Verkehr

Offered traffic = angebotener Verkehr

Traffic and its units (2)

- Unit for traffic intensity is erlang (Unit: [E])
 - ⇒ „Traffic intensity unit“, ITU-T recommendation B.18, 1993 (1 page)
 - ⇒ Mean number of occupied resources over a time period T
 - ⇒ In honour of the Danish mathematician A.K. Erlang, the founder of traffic theory
 - ⇒ The unit is dimensionless
- Traffic Volume: Total traffic during a time period T . Unit: erlang-hours (Unit: [Eh])
- Example: Call intensity is 10 calls per minute, mean service time is 3 minutes. Then, the offered traffic amounts to 30 erlang. The traffic volume during a working day (8h) is 240 Eh

13.1.6

Note that the offered traffic in the example above is a virtual measure, since not all call attempts may be served by the system. It only says that, on average, 30 resources would be occupied if all offered traffic could be served.

Busy hour

- Traffic measurements over 15 minutes period
- Busy hour is the 60 minutes per day, during which the long-time average of call attempts is maximum
 - ⇒ Highest traffic intensity may occur at different times every day
 - ⇒ Traffic intensity is different on different days of the week (Monday is usually highest traffic)
 - ⇒ Different service types may have different characteristics (voice, modem calls)
 - ⇒ Busy hour is averaged over many days
- Dimensioning of voice networks is usually done for the busy hour

13.1.7

Terminology:

Busy hour = Hauptverkehrsstunde

Busy Hour Call Attempts (BHCA)

Time Consistent Busy Hour (TCBH): busy hour in the sense defined above, i.e. averaged over a long time

Traffic varies over a number of parameters:

- 24 hours
- Weekly
- Over the month. High intensity at the beginning
- Over the year: low traffic during holiday seasons, high traffic at start of quarter
- Depending on holidays: Usually high traffic after a banking holiday

Network Performance Measures

- Call congestion B : The fraction of call attempts which observe all servers busy (= blocking probability)
- Time congestion E : The fraction of time when all servers are busy
- Traffic congestion C : The fraction of offered traffic that is not carried, possibly despite several attempts
 - ⇒ In principle similar to call congestion, if the assumption holds that a call that is blocked, leaves the system and does not make repeated call attempts for the same service request (These retries spoil the statistics of the offered traffic!)
 - ⇒ Such systems are called LCC (Lost Calls Cleared) systems

Congestion = Überlast

Outline

- Teletraffic Concepts
- **Quality of Service – Grade of Service**
- Erlang B Formula characterisation and further analysis
- Dimensioning of loss systems
- Loss systems with full accessibility
 - ⇒ Binomial case
 - ⇒ Engset case

Definition Quality of Service

- Quality of Service (QoS) is the collective effect of service performance, which determine the degree of satisfaction of a user of the service
 - ⇒ Definition and description in ITU-T recommendation E.800
 - ⇒ Traffic performance measures (Blocking probability)
 - ⇒ But also:
 - Voice quality / data correctness
 - Voice delay (effect on echo in the line)
 - Service support performance (hotline, repairs)
 - Charging correctness
 - ...

13.1.10

Definition Grade of Service

- Grade of Service (GoS) is defined as „A number of traffic engineering variables to provide a measure of adequacy of a group of resources under specified conditions. These GoS variables may be probability of loss, dial tone delay, etc.
 - ⇒ Definition in ITU-T recommendation E.600
 - ⇒ Contains a huge number of measures
 - ⇒ Is usually defined for a single network element and its connection to the „outside world“
- The key point is to translate the QoS requirements into GoS requirements for network and network node dimensioning
 - ⇒ Allocation of portions of performance measures to single networks nodes such that end-to-end QoS can be maintained

Comparison QoS - GoS

- QoS and GoS have different viewpoints
 - ⇒ QoS takes the customer's point of view
 - ⇒ GoS takes the network's point of view
- Example:
 - ⇒ QoS parameter blocking probability of 1%, i.e. customer expects that 99 out of 100 calls reach their destination
 - ⇒ Blocking may happen at several points in the network
 - ⇒ Dimensioning of network nodes such that blocking target is reached on average in the network
 - ⇒ The QoS for individual access lines may not meet the target, even though the network in total does.
- Solution: Service Level Agreements (SLA)

13.1.12

Outline

- Teletraffic Concepts
- Quality of Service – Grade of Service
- Erlang B Formula characterisation and further analysis
- Dimensioning of loss systems
- Loss systems with full accessibility
 - ⇒ Binomial case
 - ⇒ Engset case

13.1.13

Erlang B formula

- Offered load $A = \lambda/\mu$, m servers (=outgoing lines/trunks)
- State probabilities (i customers in the system):

$$p_i = \frac{\frac{A^i}{i!}}{\sum_{k=0}^m \frac{A^k}{k!}}$$

- Blocking probability (probability that system is full, i.e. m customers in the system):

$$p_m = \frac{\frac{A^m}{m!}}{\sum_{n=0}^m \frac{A^n}{n!}} = E_m(A)$$

Traffic performance values

- Call congestion B (probability that a random call is lost) is equal to the proportion of call attempts that are blocked:

$$B = \frac{\lambda \cdot p_m}{\sum_{k=0}^m \lambda \cdot p_k} = \frac{\lambda (A^m / m!) / (\sum_{l=0}^m (A^l / l!))}{\lambda / (\sum_{l=0}^m (A^l / l!)) \sum_{k=0}^m (A^k / k!)} = p_m = E_m(A)$$

- Carried traffic Y :

$$Y = \sum_{k=1}^m k \cdot p_k = A \cdot (1 - p_m) = A \cdot (1 - E_m(A))$$

- Lost traffic A_l :

$$A_l = A - Y = A \cdot E_m(A)$$

- Traffic congestion C :

$$C = \frac{A - Y}{A} = E_m(A)$$

$$E = B = C !!$$

Poisson arrival process is the only one with this property! All other arrival processes have at least one of the values different from the others

Traffic carried by the i^{th} trunk

- Traffic carried by the i^{th} trunk is denoted as a_i (= utilisation)
- **Random hunting** (all channels are selected randomly). In this case, all channels i carry the same amount of traffic on average:

$$a_i = a = \frac{Y}{m} = \frac{A \cdot \{1 - E_m(A)\}}{m}$$

- **Ordered hunting** (=sequential hunting, i.e. channels are occupied in ascending order). The traffic carried by channel i is the difference between the carried by i channels and the traffic carried by $i - 1$ channels:

$$a_i = Y_i - Y_{i-1} = A \cdot \{E_{i-1}(A) - E_i(A)\}$$

Note that a_i is independent from the total number of channels!

Ordered hunting will become important later, when we discuss grading and overflow traffic.

Improvement function

- The improvement function denotes the increase in carried traffic, when the number of channels is increased by one from m to $m + 1$:

$$F_m(A) = Y_{m+1} - Y_m = A \cdot \{E_m(A) - E_{m+1}(A)\}$$

- We have:

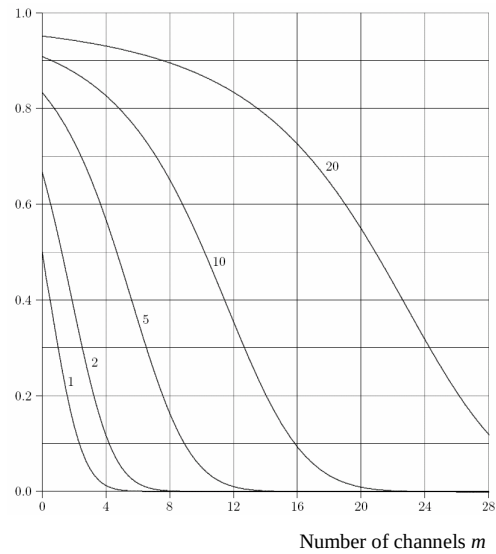
$$0 \leq F_m(A) \leq 1$$

- Boundary values can be shown to be:

$$F_0(A) = \frac{A}{1+A}, \quad F_\infty(A) = 0,$$
$$F_m(0) = 0, \quad F_m(\infty) = 1$$

Graph of improvement function

$$F_m(A) = a_{m+1} \text{ (ordered hunting)}$$



Parameter A

13.1.18

The curves show that the boundary conditions are valid.

The improvement function is an important measure for the network dimensioning under economic aspects, as we will see later.

Peakedness

- Peakedness is the ratio between the variance and the mean value of a distribution
- For the Poission system with limited number of servers, we get:

$$Z = \frac{\sigma^2}{E\{i\}} = 1 - A\{E_{m-1}(A) - E_m(A)\} = 1 - a_m \quad (\text{ordered hunting})$$

The unit is [number of channels]

13.1.19

Peakedness = Spitzigkeit

Numerical evaluation of Erlang B formula

- The problem with numerical evaluation is mainly the faculty in the denominators which lead to memory overflow:

$$E_m(A) = \frac{\frac{A^m}{m!}}{\sum_{n=0}^m \frac{A^n}{n!}}$$

- The problem with numerical evaluation is mainly the faculty in the denominators which lead to memory overflow. Hence, calculation is usually performed recursively:

$$E_x(A) = \frac{A \cdot E_{x-1}(A)}{x + A \cdot E_{x-1}(A)}, \quad E_0(A) = 1$$

- The recursive calculation of the inverse loss probability is the numerically most stable method:

$$\frac{1}{E_x(A)} = I_x(A) = 1 + \frac{x}{A} I_{x-1}(A), \quad I_0(A) = 1$$

13.1.20

The recursive calculation of $I_x(A)$ is exact and even for large values of m and A , it does not exhibit rounding errors

Outline

- Teletraffic Concepts
- Quality of Service – Grade of Service
- Erlang B Formula characterisation and further analysis
- **Dimensioning of loss systems**
- Loss systems with full accessibility
 - ⇒ Binomial case
 - ⇒ Engset case

13.1.21

Fixed Blocking Probability

- General assumption: Blocking probability should be below a given value, e.g. 1%
- Remember trunking gain: The higher the number of channels, the higher the possible utilisation of channel capacity for a fixed blocking rate

m	1	2	5	10	20	50	100
$A (E = 1\%)$	0.010	0.153	1.361	4.461	12.031	37.901	84.064
a	0.010	0.076	0.269	0.442	0.596	0.750	0.832
$F_m(A)$	0.000	0.001	0.011	0.027	0.052	0.099	0.147
$A_1 = 1.2 \cdot A$	0.012	0.183	1.633	5.353	14.437	45.482	100.877
$E [\%]$	1.198	1.396	1.903	2.575	3.640	5.848	8.077
a	0.012	0.090	0.320	0.522	0.696	0.856	0.927
$F_m(A_1)$	0.000	0.002	0.023	0.072	0.173	0.405	0.617

13.1.22

Two observations from the table above:

1. The utilisation a (for random hunting) increases with increasing numbers of resources (this is the trunking gain)
2. Large channel groups are more sensitive to an increase in offered traffic. This effect can be seen in the lower part of the table. The achieved blocking probability increases massively with increasing group size.

Improvement principle

- Replacing the required loss probability by a more economically inspired parameter leads to using the improvement function $F_m(A)$ and dimensioning the network based on a target value for $F_m(A)$ ($F_B = 0.05$ in table below)

m	1	2	5	10	20	50	100
$A (F_B = 0.05)$	0.271	0.607	2.009	4.991	11.98	35.80	78.73
a	0.213	0.272	0.387	0.490	0.593	0.713	0.785
$E_{1,m}(A) [\%]$	21.29	10.28	3.72	1.82	0.97	0.47	0.29
$A_1 = 1.2 \cdot A$	0.325	0.728	2.411	5.989	14.38	42.96	94.476
$E \{ \%$	24.51	13.30	6.32	4.28	3.55	3.73	4.62
a	0.245	0.316	0.452	0.573	0.693	0.827	0.901
$F_m(A_1)$	0.067	0.074	0.093	0.120	0.169	0.294	0.452

13.1.23

The blocking probability of small groups becomes better by increasing at the same time the blocking probability. On the other hand, the increasing offered traffic has a less severe impact than in the previous example.

The improvement function is equal to the marginal increase of carried traffic for a marginal increase in channel capacity. The dimensioning with this criterion leads to equal gradients of the improvement function for all numbers of channels.

Economic calculation

- Economical model of the improvement principle:
 - Observe a fixed time unit
 - Income per carried erlang (increases with increasing utilisation): g
 - The cost of a cable depends on the number of channels m and is given by:

$$C_m = C_0 + c \cdot m$$

- The total cost for a given number of channels is then (a) cost of cable and (b) cost due to lost traffic (=missing income):

$$C_m = g \cdot A \cdot E_m(A) + C_0 + c \cdot m$$

- We are searching for a minimum of the cost function (with respect to m):

$$C_{m-1} > C_m \cap C_{m+1} > C_m;$$

$$A \cdot \{E_{n-1}(A) - E_n(A)\} > \frac{c}{g} \geq A \cdot \{E_n(A) - E_{n+1}(A)\};$$

$$F_{m-1}(A) > F_B \geq F_m(A) \quad \text{where} \quad F_B = \frac{c}{g} = \frac{\text{cost per extra channel}}{\text{income per extra channel}}$$

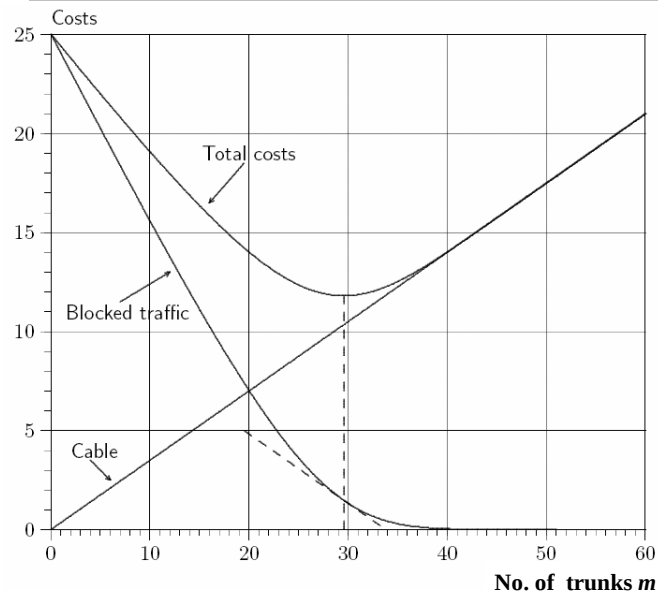
13.1.24

The minimisation of the costs above is of course for an integer number of channels.

Es gilt:

$$F_m(A) = A \cdot \{E_m(A) - E_{m+1}(A)\}$$

Graphical representation



13.1.25

Outline

- Teletraffic Concepts
- Quality of Service – Grade of Service
- Erlang B Formula characterisation and further analysis
- Dimensioning of loss systems
- Loss systems with full accessibility
 - ⇒ Binomial case
 - ⇒ Engset case

Introduction

- Assumptions
 - ⇒ Full accessability, i.e. any free server can be accessed by any arriving customer
 - ⇒ Lost Calls Cleared, i.e. blocked calls disappear from the system (no immediate retry!)
- If the number of customers is limited, the arrival process is influenced by the customers that are undergoing service
 - ⇒ Number of potential customers S
 - ⇒ Poisson case: $\lim_{S \rightarrow \infty, \gamma \rightarrow 0} S \cdot \gamma = \lambda \Rightarrow (S \pm i) \cdot \gamma \approx S \cdot \gamma = \lambda$
 - ⇒ Engset case: $S \cdot \gamma = \lambda \Rightarrow (S-i) \cdot \gamma \neq \lambda$
 - ⇒ State-dependent Poisson arrival process

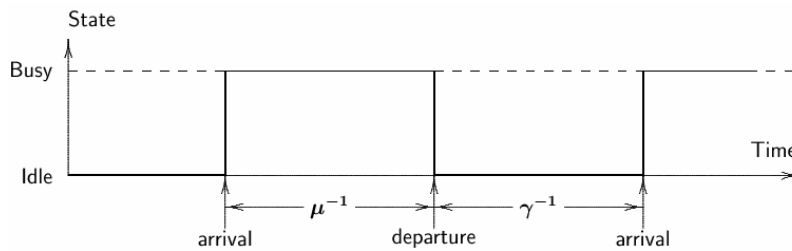
13.1.27

Binomial case

- Number of servers n , number of potential customers S

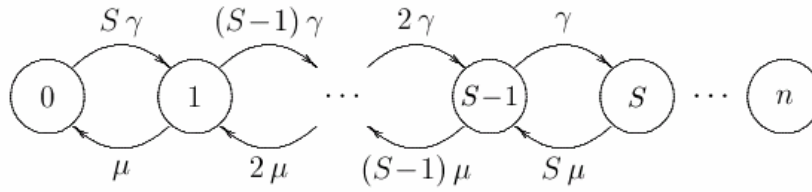
$$n \geq S$$

- This means that no call is lost
- Individual customers switch between the states „busy“ and „idle“ with times between service requests γ^{-1} and service rate μ



13.1.28

Binomial case Equilibrium equations



$$\begin{aligned}
 S \cdot \gamma \cdot p(0) &= \mu \cdot p(1) \\
 (S-1) \cdot \gamma \cdot p(1) &= 2\mu \cdot p(2) \\
 &\dots \\
 (S-i) \cdot \gamma \cdot p(i) &= (i+1)\mu \cdot p(i+1) \\
 &\dots \\
 \gamma \cdot p(S-1) &= S\mu \cdot p(S)
 \end{aligned}$$

Binomial case Equilibrium equations (2)

- All state probabilities can be expressed by $p(0)$:

$$p(1) = \frac{S \cdot \gamma}{\mu} \cdot p(0) = p(0) \cdot \binom{S}{1} \cdot \left(\frac{\gamma}{\mu}\right)^1$$

$$p(2) = \frac{(S-1) \cdot \gamma}{2\mu} \cdot p(1) = p(0) \cdot \binom{S}{2} \cdot \left(\frac{\gamma}{\mu}\right)^2$$

...

$$p(i) = \frac{(S-i+1) \cdot \gamma}{i \cdot \mu} \cdot p(i-1) = p(0) \cdot \binom{S}{i} \cdot \left(\frac{\gamma}{\mu}\right)^i$$

...

$$p(S) = \frac{\gamma}{S \cdot \mu} \cdot p(S-1) = p(0) \cdot \binom{S}{S} \cdot \left(\frac{\gamma}{\mu}\right)^S$$

Binomial case Equilibrium equations (3)

$p(0)$ is derived by computing the sum of all probabilities, which needs to equal 1:

$$\begin{aligned} 1 &= p(0) \left\{ \binom{S}{1} \cdot \left(\frac{\gamma}{\mu}\right) + \binom{S}{2} \cdot \left(\frac{\gamma}{\mu}\right)^2 + \dots + \binom{S}{S} \cdot \left(\frac{\gamma}{\mu}\right)^S \right\} = \\ &= p(0) \cdot \left\{ 1 + \frac{\gamma}{\mu} \right\}^S \end{aligned}$$

with the help of the binomial expansion.

By letting $\beta = \gamma/\mu$ we obtain

$$p(0) = \frac{1}{(1 + \beta)^S}$$

13.1.31

The parameter β is the offered traffic per idle source, whereas the offered traffic of a busy source is zero.

Binomial case Probability distribution

We define

$$\alpha = \frac{\beta}{1+\beta} = \frac{\gamma}{\mu + \gamma}$$

Then

$$\begin{aligned} p(i) &= \binom{S}{i} \cdot \beta^i \cdot \frac{1}{(1+\beta)^S} \\ &= \binom{S}{i} \cdot \left(\frac{\beta}{1+\beta} \right)^i \cdot \left(\frac{1}{1+\beta} \right)^{S-i} = \\ &= \binom{S}{i} \cdot \alpha^i \cdot (1-\alpha)^{S-i}; \quad i = 0, 1, \dots, S; \quad 0 \leq S \leq n \end{aligned}$$

13.1.32

α is equivalent to the carried traffic a per source, given the fact that a service request is never blocked.

The last formula above is the binomial distribution. This is the reason why this case is called the binomial case.

Binomial case Traffic characterisation

- β is the offered traffic per idle source.
 - The offered traffic per source is different and depends on the congestion of the system (the more congestion, the more idle time)
 - By definition, the offered traffic is equal to the carried traffic in a system with no congestion

$$\alpha = \frac{\beta}{1 + \beta} = \text{offered traffic per source} = \frac{\gamma}{\mu + \gamma},$$

$$A = S \cdot \alpha = \text{total offered traffic},$$

$$a = \text{carried traffic per source},$$

$$Y = S \cdot a = \text{carried traffic} = A = S \cdot \alpha$$

13.1.33

Regarding the offered traffic per source, remember the graphic on slide 28, where the relation between γ and μ is shown. There, γ represents the mean arrival rate of an idle source, whereas it is not active when it is being served. The mean service duration is given by μ^{-1} , so $\gamma + \mu$ somehow represents the overall time, whereas γ stands for the arrivals when idle.

Binomial case Traffic characterisation (2)

Time congestion : $E = 0$ $S < n$

$$E = p(n) = \alpha^n \quad S = n$$

Carried traffic : $Y = S \cdot a = \sum_{i=0}^S i \cdot p(i) = S \cdot \alpha = A$

Traffic congestion : $C = \frac{A-Y}{Y} = 0$

Peakedness : $Z = \frac{\sigma^2}{E\{i\}} = \frac{S \cdot \alpha \cdot (1-\alpha)}{S \cdot \alpha} = 1 - \alpha = \frac{1}{1 + \beta} < 1$

Mean Time spent in state i : $\tau(i) = 1 / [(S-i) \cdot \lambda + i \cdot \mu]$
(exponentially distributed)

A peakedness of less than 1 is denoted as „smooth traffic“

Outline

- Teletraffic Concepts
- Quality of Service – Grade of Service
- Erlang B Formula characterisation and further analysis
- Dimensioning of loss systems
- Loss systems with full accessibility
 - ⇒ Binomial case
 - ⇒ Engset case

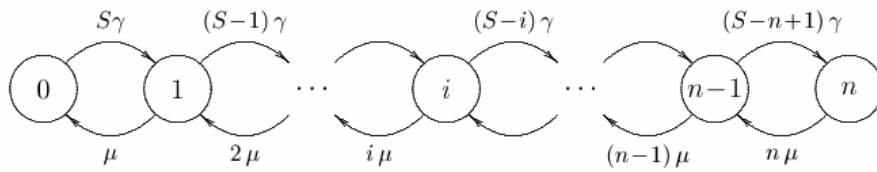
Engset case

- Number of servers n , number of potential customers S

$$n \geq S$$

- This means that call can get lost, when an arriving customer finds the system fully occupied
- Individual customers switch between the states „busy“ and „idle“ with times between service requests γ^{-1} and service rate μ

Engset case Equilibrium equations



$$S \cdot \gamma \cdot p(0) = \mu \cdot p(1)$$

$$(S-1) \cdot \gamma \cdot p(1) = 2\mu \cdot p(2)$$

...

$$(S-i) \cdot \gamma \cdot p(i) = (i+1)\mu \cdot p(i+1)$$

...

$$(S-n+1) \cdot \gamma \cdot p(n-1) = n\mu \cdot p(n)$$

13.1.37

The markov chain is truncated at $i=n$.

Engset case Probability distribution

The normalisation over the sum of all probabilities results in :

$$1 = p(0) \cdot \left\{ 1 + \binom{S}{1} \cdot \left(\frac{\gamma}{\mu} \right) + \dots + \binom{S}{n} \cdot \left(\frac{\gamma}{\mu} \right)^n \right\}$$

with $\beta = \gamma / \mu$, we obtain the state probabilities :

$$p(i) = \frac{\binom{S}{i} \cdot \beta^i}{\sum_{j=0}^n \binom{S}{j} \cdot \beta^j} = \frac{\binom{S}{i} \cdot \alpha^i (1-\alpha)^{S-i}}{\sum_{j=0}^n \binom{S}{j} \cdot \alpha^j (1-\alpha)^{S-j}}, \text{ where } \alpha = \frac{\beta}{1+\beta} \Rightarrow \beta = \frac{\alpha}{1-\alpha}$$

This formula is called Engset distribution. Due to the second form above, it is often denoted as the truncated binomial distribution.

The Norwegian T. Engset was the first one to publish the model with a final number of resources in 1918.

Engset case Traffic characterisation

- The Engset case is more difficult to handle than the Erlang loss system
- Important parameters: $\beta = \gamma/\mu$ offered traffic per idle source, S, n
- Time congestion E (Proportion of time the system is blocked for new service requests)

$$E_{n,S}(\beta) = p(n) = \frac{\binom{S}{n} \cdot \beta^n}{\sum_{j=0}^n \binom{S}{j} \cdot \beta^j}; \quad S \geq n$$

Engset case Traffic characterisation (2)

- Call congestion B (Proportion of service requests that find the system busy and are, therefore, blocked.) During one unit of time, we observe the ratio between the number of blocked call attempts and the total number of call attempts:

$$B_{n,s} = \frac{p(n) \cdot (S-n) \cdot \gamma}{\sum_{j=0}^n (p(j) \cdot (S-j) \cdot \gamma)} = \frac{\binom{S}{n} \cdot \beta^n \cdot (S-n) \cdot \gamma}{\sum_{j=0}^n \left(\binom{S}{j} \cdot \beta^j \cdot (S-j) \cdot \gamma \right)}$$

$$\text{Using } \binom{S}{i} \cdot \frac{S-i}{S} = \binom{S-1}{i} \Rightarrow \binom{S}{i} \cdot (S-i) = \binom{S}{i} \frac{S-i}{S} \cdot S = \binom{S-1}{i} \cdot S$$

$$\text{we obtain : } B_{n,s}(\beta) = \frac{\binom{S-1}{n} \cdot \beta^n}{\sum_{j=0}^n \left(\binom{S-1}{j} \cdot \beta^j \right)} = E_{n,S-1}(\beta)$$

13.1.40

The step in between is:

$$\begin{aligned} B_{n,s} &= \frac{\binom{S}{n} \cdot \beta^n \cdot (S-n) \cdot \gamma}{\sum_{j=0}^n \left(\binom{S}{j} \cdot \beta^j \cdot (S-j) \cdot \gamma \right)} = \\ &= \frac{S \cdot \binom{S-1}{n} \cdot \beta^n \cdot \gamma}{\gamma \cdot S \cdot \sum_{j=0}^n \left(\binom{S-1}{j} \cdot \beta^j \right)} = \\ &= \frac{\binom{S-1}{n} \cdot \beta^n}{\sum_{j=0}^n \left(\binom{S-1}{j} \cdot \beta^j \right)} \end{aligned}$$

We can interpret this result such that the probability that a call attempt from a random source is rejected is equal to the probability that the remaining $(S-1)$ sources occupy all n channels. This is called the „arrival theorem“

Engset case Traffic characterisation (3)

- **Arrival Theorem:**

For all systems with a limited number of sources, a random source upon arrival will observe the state of the system as if the source itself does not belong to the system

- **Carried traffic (mean value of served call attempts):**

Using cut equations $(S - i + 1) \cdot \gamma \cdot p(i - 1) = i \cdot \mu \cdot p(i)$ and substituting $j = i - 1$, we get

$$Y = \sum_{i=1}^n i \cdot p(i) = \sum_{i=1}^n \frac{\gamma}{\mu} \cdot (S - i + 1) \cdot p(i - 1) =$$

$$= \sum_{j=0}^{n-1} \beta \cdot (S - j) \cdot p(j) = \sum_{j=0}^n \beta \cdot (S - j) \cdot p(j) - \beta \cdot (S - n) \cdot p(n)$$

$$Y = \beta \cdot (S - Y) - \beta \cdot (S - n) \cdot E \quad \text{with } E = E_{n,S} = p(n);$$

finally

$$Y = \frac{\beta}{1 + \beta} \{S - (S - n) \cdot E\}$$

Engset case Traffic characterisation (3)

- **Traffic congestion C** (The relative share of the offered traffic that is not served)

$$C = \frac{A - Y}{A} = \frac{S \cdot \frac{\beta}{1+\beta} - \frac{\beta}{1+\beta} \cdot (S - (S - n) \cdot E)}{S \cdot \frac{\beta}{1+\beta}}$$

$$C_{n,S} = \frac{S - n}{S} \cdot E_{n,S}$$

- **Number of call attempts per time unit Λ**

$$\Lambda = \sum_{i=0}^n p(i) \cdot (S - i) \cdot \gamma = S \cdot \gamma - Y \cdot \gamma = (S - Y) \cdot \gamma$$

This means that $(S - Y)$ is the mean number of idle sources

For calculating A , see slide 33. Note that the offered traffic assumes that no blocking occurs, i.e. the offered load calculated for the binomial case is the same as the offered load in the Engset case!

Engset case Traffic characterisation (4)

- **Lost traffic A_l**

$$\begin{aligned} A_l &= A \cdot C \\ &= S \cdot \frac{\beta}{1 + \beta} \cdot \frac{S - n}{S} \cdot E \\ &= \frac{(S - n)\beta}{1 + \beta} E \end{aligned}$$

Engset case Evaluation of Engset formula (1)

- Recursion in n :**

Using $\beta = \gamma/\mu$, we get:

$$\begin{aligned} E_{x,S} &= \frac{(S-x+1)\gamma \cdot E_{x-1,S}(\beta)}{x\mu + (S-x+1)\gamma \cdot E_{x-1,S}(\beta)} = \\ &= \frac{(S-x+1)\beta \cdot E_{x-1,S}(\beta)}{x + (S-x+1)\beta \cdot E_{x-1,S}(\beta)}; \\ E_{0,S}(\beta) &= 1 \end{aligned}$$

Engset case Evaluation of Engset formula (2)

- **Recursion in S:**

$$E_{n,S}(\beta) = \frac{S}{S-n} \cdot \frac{E_{n,S-1}(\beta)}{1 + \beta(1 - E_{n,S-1}(\beta))}, \quad S > n$$

$$E_{n,n}(\beta) = \alpha^n$$

- **Recursion in S and n:**

$$E_{n,S}(\beta) = \frac{S\alpha \cdot E_{n-1,S-1}(\beta)}{n + (S-n)\alpha \cdot E_{n-1,S-1}(\beta)};$$

$$E_{0,S-n}(\beta) = 1$$

13.1.45

For the initial value $E_{n,n}(\beta)$ of the recursion in S see slide 34.

The recursion formula is obtained by inserting the recursion formula for n into the recursion formula for S.

For the numerical stability of the recursion formulae, see corresponding literature (e.g. ITU-D Study Group 2, „Handbook Teletraffic Engineering“, Geneva, December 2002)

Discussion of Engset case

Comparison Engset case and Poisson case

Poisson case

$$E_m(A) = \frac{A^m}{m!} \cdot \frac{m!}{\sum_{n=0}^m \frac{A^n}{n!}}$$

$$B = E_m(A)$$

$$C = E_m(A)$$

Engset case

$$E_{n,S}(\beta) = \frac{\binom{S}{n} \cdot \beta^n}{\sum_{j=0}^n \binom{S}{j} \cdot \beta^j}$$

$$B = E_{n,S-1}(\beta)$$

$$C = \frac{S-n}{S} \cdot E_{n,S}(\beta)$$

13.1.46

In the Poisson-case, we have that $E = B = C$. As stated earlier, it is the only system where this is valid (see slide 15). This means that the system can be characterised by a single value, the loss parameter.

We have for the Engset case $E \neq B \neq C$, so it is necessary to think about the detailed meaning of each of the parameters.

Engset case Relation of Parameters

Relation between E , B and C :

$$B = E_{n,S-1} \Rightarrow E > B$$

$$E = E_{n,S} = \frac{S}{S-n} \cdot \frac{B}{1+\beta(1-B)}; \quad B = \frac{(S-n) \cdot E \cdot (1+\beta)}{S + (S-n) \cdot E \cdot \beta}$$

$$C = \frac{S-n}{S} \cdot E \Rightarrow E = \frac{S}{S-n} \cdot C$$

Using the result from the previous equation yields

$$C = \frac{B}{1+\beta(1-B)} \Rightarrow B = \frac{(1+\beta)C}{1+\beta \cdot C}$$

$$\Rightarrow E > B > C$$

13.1.47

It is immediately obvious that, having one source less results in a smaller time congestion E , i.e. $E_{n,S-1} < E_{n,S}$ and, hence, $E > B$.

The relation between E and B exploits the equation for B in dependence of $E_{n,S-1}$ on slide 39 and the iteration formula for the engset formula for S on slide 44.

The last relation is based on the fact that all parameters as well as β are between zero and one

Engset case Interpretation of parameters

Questions: Who is concerned by each parameter? (Subscriber/provider)?
Why do the parameters have different values?

- **Time congestion E** (Proportion of time the system is blocked for new service requests)
- **Call congestion B** (Proportion of service requests that find the system busy and are, therefore, blocked.)
- **Traffic congestion C** (The relative share of the offered traffic that is not served, possibly despite several attempts)

Engset case Interpretation of parameters

Questions: Who is concerned by each parameter? (Subscriber/provider)?
Why do the parameters have different values?

- **Time congestion E** (Proportion of time the system is blocked for new service requests)
 - Is only a virtual measure which can be used to derive the remaining parameters
- **Call congestion B** (Proportion of service requests that find the system busy and are, therefore, blocked.)
 - Concerns the subscriber
- **Traffic congestion C** (The relative share of the offered traffic that is not served, possibly despite several attempts)
 - This is the lost profit of the service provider

13.1.49

The reasons for the different values of the parameters are:

- The time congestion represents the time during which the system is busy. Due to the fact that n subscribers are busy and this is at the same time the maximum number of subscribers that can be served, the arrival rate is at its lowest possible value. This is the reason why the number of potential call attempts during such periods is small and, hence, the proportion of blocked calls is smaller than the time share, i.e. $E > B$.
- The call congestion considers all call attempts, including retries. The traffic congestion considers retries as belonging to the same conversation (Remember: Offered load is the limit of the call rate for $n \rightarrow \infty$). So if the same conversation request is blocked more than once, it is accounted for only once in C , whereas each attempt is counted in the case of B . In other words, if blocking occurs, the call attempt intensity is higher than without blocking. This is why $B > C$.

Engset case Example 1

Consider an Engset system with $n=3$, $S=4$, $\gamma=1/3$ calls per time unit and $1/\mu = 1$ time unit. Find β , α , A , Z , E , B and C .

$$\beta = \frac{\gamma}{\mu} = \frac{1}{3} \text{ erlang (offered traffic per idle source)}$$

$$\alpha = \frac{\beta}{1+\beta} = \frac{1}{4} \text{ erlang (offered traffic per source)}$$

$$A = S \cdot \alpha = 1 \text{ erlang (offered traffic)}$$

$$Z = 1 - A/S = 3/4 \text{ (Peakedness)}$$

Engset case Example 1 (2)

Consider an Engset system with $n=3$, $S=4$, $\gamma=1/3$ calls per time unit and $1/\mu = 1$ time unit. Find β , α , A , Z , E , B and C .

State i	Arrival rate $\gamma(i)$	Service rate $\mu(i)$	State prob. $p(i)$	$i \cdot p(i)$
0	4/3	0	0,3176	0,0000
1	3/3	1	0,4235	0,4235
2	2/3	2	0,2118	0,4235
3	1/3	3	0,0471	0,1412
Total			1	0,9882

$= Y$

$$E = E_{3,4}(1/3) = p(3) = 0,0471 \text{ (Time congestion)}$$

$$B = \frac{\gamma(3) \cdot p(3)}{\sum_{i=0}^3 \gamma(i) \cdot p(i)} = \frac{0,0157}{1,0039} = 0,0156 \text{ (Call congestion)}$$

$$E > B > C !$$

$$C = \frac{A - Y}{Y} = \frac{1 - 0,9882}{1} = 0,0118 \text{ (Traffic congestion)}$$

13.1.51